B.Tech.

First Semester Examination, 2009-2010

Mathematics-I (101-F)

Note : Attempt *five* questions in total. Question 1 is compulsory.

Q. 1. (Compulsory Question)

(a) Test the convergence of divergence of the series : $\Sigma \cot^{-1} n^2$.

Ans. Let $$\Sigma u_n = \Sigma \cot^{-1} n^2$$

Then, $$u_n = \cot^{-1} n^2 = \tan^{-1} \frac{1}{n^2}$$

$$= \frac{1}{n^2} - \frac{1}{3n^6} + \frac{1}{5n^{10}} - \ldots\ldots \infty$$

$$= \frac{1}{n^2}\left[1 - \frac{1}{3n^4} + \ldots\ldots\ldots\ldots \infty\right]$$

Let $$V_n = \frac{1}{n^2}$$

Then, $$\frac{u_n}{v_n} = 1 - \frac{1}{3ny} + \ldots\ldots$$

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = 1 \text{ (finite and non-zero)}$$

By comparison test, Σu_n and Σv_n behave alike.

But $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent $(\because p = 2 > 1)$.

Hence given series is also convergent.

Q. 1. (b) Discuss the convergence of the series : $x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots$.

Ans. Here, $$u_n = n.x^n$$

$$u_{n+1} = (n+1)x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{n.x^n}{(n+1)x^{n+1}} = \frac{n.x}{n\left(1+\frac{1}{x}\right).x^n x}$$

$$\underset{n\to\infty}{Lt}\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{n}\right)x} = \frac{1}{x}$$

By D' Alembort ratio test.

Σu_n is convergent if $\frac{1}{x} > 1$ i.e., $x < 1$ and divergent if $\frac{1}{x} < 1$ i.e., $x > 1$.

At $x = 1$ $\qquad u_n = n$

$$\underset{n\to\infty}{Lt}\ u_n = \infty$$

Σu_n divergent in this case.

Hence given series is convergent if $x < 1$ and divergent $x \geq 1$.

Q. 1. (c) If $A = \begin{bmatrix} -2 & 5 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$, find eigen values of A^5.

Ans. Given, $\qquad A = \begin{bmatrix} -2 & 5 & 1 \\ - & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

Eigen values of A are $-2, 2, 3$

Then eigen values of A^5 are $(-2)^5, (2)^5, (3)^5$

$$-32, 32, 243$$

Q. 1. (d) Write down the quadratic form corresponding to the matrix : $\begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$.

Ans. Let $\qquad A = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$

Here, $\qquad a = 2, b = 3, c = 3,$

$$h = 2,\ g = \frac{5}{2},\ f = \frac{1}{2}$$

Their quadrative form is.

$$q = ax^2 + by^2 + cz^2 + 2gzx + 2hxy + 2fyz$$

$$q = 2x^2 + 3y^2 + 3z^2 + 5zx + 4xy + yz.$$

Q. 1. (e) Find the nth derivative of $\frac{1}{1-5x+6x^2}$.

Ans. Now,
$$\frac{1}{1-5x+6x^2} = \frac{1}{6x^2-5x+1} = \frac{1}{(3x+1)(2x-1)}$$

$$\frac{1}{6x^2-5x+1} = \frac{\frac{2}{5}}{2x-1} - \frac{\frac{3}{5}}{3x+1}$$

Let
$$y = \frac{2}{5}\cdot\frac{1}{2x-1} - \frac{3}{5}\cdot\frac{1}{3x+1}$$

Diff. in times const of.

$$y_n = \frac{2}{5}\frac{(-1)^n\, n!2^n}{(2n-1)^{n+1}} - \frac{3}{5}\cdot\frac{(-1)^n\, n!3^n}{(3n+1)^{n+1}}$$

$$\left[\begin{array}{l} \because y = \dfrac{1}{an+s} \\ y_n = \dfrac{(-1)^n . n!}{(an+x)} \end{array}\right.$$

$$y_n = \frac{(-1)^n . n!}{5}\left[\frac{2^{n+1}}{(2n-1)^{n+1}} - \frac{3^{n+1}}{(3n+1)^{n+1}}\right].$$

Q. 1. (f) If $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$ show that: $\frac{x\partial u}{\partial x} + \frac{y\,\partial u}{\partial z} + \frac{z\partial u}{\partial z} = 3\tan u$.

Ans. Given,
$$u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$$

Then
$$\sin u = \frac{x\left[1+2(y/x)+3(z/x)\right]}{x^4\sqrt{1+\left(\frac{y}{x}\right)^8+\left(\frac{z}{x}\right)^8}} = x^{-3}\phi\left(\frac{y}{x}, \frac{z}{x}\right)$$

So, sin u is a homogeneous function is x, y, z of degree (–3).
Than by Euler's theorem,

$$x.\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) + z.\frac{\partial}{\partial z}(\sin u) = -3\sin u$$

$$x.\cos u.\frac{\partial u}{\partial x}+y.\cos u.\frac{\partial u}{\partial y}+\cos u\frac{\partial u}{\partial z}=-3\sin u$$

$$x.\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z.\frac{\partial u}{\partial z}=-3\frac{\sin u}{\cos u}$$

$$x.\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z.\frac{\partial u}{\partial z}=-3\tan u$$

$$x.\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z.\frac{\partial u}{\partial z}9+3\tan u=0$$

Q. 1. (g) Find by double integration, the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Ans. Given :

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$$

$$\frac{x^2}{a^2}\le 1-\frac{y^2}{b^2}$$

$$x^2\le a^2\left(1-\frac{y^2}{b^2}\right)$$

$$-a\sqrt{1-\frac{y^2}{b^2}}\le x\le a\sqrt{1-\frac{y^2}{b^2}}$$

Where $y^2\le b^2$ i.e., $-b\le y\le b$

$\therefore$ **Area** of ellipse

$$=\iint\limits_R dx\ dy$$

$$=\int_{-a}^{b}\int_{-a\sqrt{1-\frac{y^2}{b^2}}}^{a\sqrt{1-\frac{y^2}{b^2}}}dx\,dy=\int_0^b\int_0^{a\sqrt{1-\frac{y^2}{b^2}}}2\,dx\,dy$$

$$=\int_0^b[2x]_0^{a\sqrt{1-\frac{y^2}{b^2}}}dy=\int_{-b}^{b}2a.\sqrt{1-\frac{y^2}{s^2}}\,dy$$

$$= \int_0^b 4a.\sqrt{1-\frac{y^2}{b^2}}\,dy = \frac{4a}{b}\int_0^b \sqrt{b^2-y^2}\,dy$$

$$= \frac{4a}{b}\left[\frac{y\sqrt{b^2-y^2}}{2}+\frac{b^2}{2}\sin^{-1}\frac{y}{b}\right]_0^b$$

$$= \frac{4a}{b}\left(\frac{52}{2}\sin^{-1}1\right) = 2a\left(\frac{\pi}{2}\right)^b$$

$$= \pi ab\,.$$

Q. 1. (h) Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$.

Ans. Let

$$I = \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[(x+z).y+\frac{y^2}{2}\right]_{x-z}^{x+z}dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left\{(x+z)(x+z)+\frac{(x+z)^2}{2}\right\}-\left\{(x+z).(x-z)+\frac{(x-z)^2}{2}\right\}dx\,dy$$

$$= \int_{-1}^{1}\int_{0}^{z}\frac{3(x+z)^2}{2}-\left\{x^2-z^2+\frac{x^2+z^2+2xz}{2}\right\}dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[\frac{3}{2}(x+z)^2-\left\{\frac{2x^2-2z^2+x^2+z^2-2xz}{2}\right\}\right]dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[\frac{3}{2}(x+z)^2-\frac{1}{2}\left(3x^2-z^2-2xz\right)\right]dx\,dz$$

$$= \frac{1}{2}\int_{-1}^{1}\int_{0}^{z}\left[3.(x+z)^2-3x^2+z^2+2xz\right]dx\,dz$$

$$= \frac{1}{2}\int_{-1}^{1}3\left[\frac{(x+z)^3}{3}\right]_0^z-\left(\frac{3x^3}{3}\right)_0^z+\left(z^2.x+x^2z\right)_0^z\,dz$$

$$= \frac{1}{2}\int_{-1}^{1}\left[(2z)^2 - z^3 - z^3 + z^3 + z^3\, dz\right]$$

$$= \frac{1}{2}\int_{-1}^{1} 8z^3 - z^3 + z^3\, dz = 0 \qquad \left[\because z^3 \text{ is odd function}\right]$$

Section-A

Q. 2. (a) Prove that if series Σu_n is convergent than $u_n \to 0$ as $n \to \infty$. Is the converse true? Illustrate with an example.

Ans. Let S_n denotes the partial sum of the first n terms of the series $\sum_{n=1}^{\infty} u_n$

$$\therefore \qquad S_n = u_1 + u_2 + \ldots\ldots + u_{n-1} + u_n$$

$$S_{n-1} = u_1 + u_2 + \ldots\ldots + u_{n-1}$$

$$\therefore \qquad S_n - S_{n-1} = u_n \qquad \ldots(1)$$

Let $\sum_{n=1}^{\infty} u_n$ be convergent to S.

Therefore, $\lim_{n\to\infty} S_n = S$ and $S_{n-1} = S$

From equation (1) $\quad u_n = S_n - S_{n-1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

Thus, we have if $\sum_{n=1}^{\infty} u_n$ converges, then

$$\lim_{n\to\infty} u_n = 0$$

But the converse is not true. For example consider the series.

$$\Sigma u_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n} + \ldots\ldots$$

Here, $\quad u_n = \frac{1}{n}$

$$\therefore \qquad \lim_{n\to\infty} u_n = 0$$

Suppose that $\sum_{n=1}^{\infty} u_n$ is also convergent.

$\therefore$ By Cauchy's general principle of convergence. For given $\in > 0$ there exists a positive integer m such that

$$|S_n - S_m| < \in \text{ for all } n > m.$$

$$|u_{m+1} + u_{m+2} + \dots + u_n| < \in \text{ for all } n > m.$$

Taking $n = 2m$, we get

$$|u_{m+1} + u_{m+2} + \dots + u_{2m}| < \in$$

$$\left|\frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{2m}\right| < \in \qquad \dots(1)$$

Now,

$$\left|\frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{2m}\right| > \frac{1}{2m} + \frac{1}{2m} + \dots + \frac{1}{2m} = \frac{m}{2m} = \frac{1}{2}$$

Taking, $\in = \frac{1}{2}$, we get

$$\left|\frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{2m}\right| > \in \qquad \dots(2)$$

From (1) and (2), we get a contradiction. Hence, $\sum_{n=1}^{\infty} u_n$ is not convergent even if $\lim_{n\to\infty} u_n = 0$.

Q. 2. (b) Discuss the convergence of the series : $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} (p > 0)$.

Ans. Here

$$u_n = \frac{1}{n(\log n)^p} = f(n) \qquad \therefore \qquad f(x) = \frac{1}{x(\log x)^p}$$

For $x \geq 2$, $p > 0$, f(x) is +ve and monotonic decreasing.

$\therefore$ By Cauchy's Integral Test $\sum_{n=2}^{\infty} u_n$ and $\int_2^{\infty} f(x)\,dx$ converge or diverge together.

Case (1) When $p \neq 1$

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} (\log x)^{-p} \cdot \frac{1}{x}\,dx = \left[\frac{(\log x)^{-p+1}}{-p+1}\right]_2^{\infty}$$

Subcase (1) when $p > 1$, p–1 **+ve, so that**

$$\int_2^{\infty} f(x)\,dx = -\frac{1}{p-1}\left[\frac{1}{(\log x)^{p-1}}\right]_2^{\infty}$$

$$= -\frac{1}{p-1}\left[0 - \frac{1}{(\log 2)^{p-1}}\right] = \frac{1}{(p-1)(\log 2)^{p-1}} = \text{finite}$$

$$\Rightarrow \qquad \int_2^{\infty} f(x)\,dx \text{ converges} \Rightarrow \sum_{n=2}^{\infty} u_n \text{ converges}$$

Sub case (2) when $p < 1$, $1-p$ is +ve, so that

$$\int_2^{\infty} f(x)\,dx = \frac{1}{1-p}\left[(\log x)^{1-p}\right]_2^{\infty} = \frac{1}{1-p}\left[\infty - (\log 2)^{-p}\right] = \infty$$

$$\Rightarrow \qquad \int_2^{\infty} f(x)\,dx \text{ diverges} \Rightarrow \sum_{n=2}^{\infty} u_n \text{ diverges}$$

Case (II) : When $p = 1$, $f(x) = \dfrac{1}{x\log x}$

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{dx}{x\log x}\,dx = \left[\log\log x\right]_2^{\infty} = \infty - \log\log 2 = \infty$$

$$\Rightarrow \qquad \int_2^{\infty} f(x)\,dx \text{ divergence} \Rightarrow \sum_{n=2}^{\infty} u_n \text{ diverges.}$$

Hence, Σu_n converges if $p > 1$ and diverges if $0 < p \le 1$.

Q. 3. (a) Test for convergence the series : $\dfrac{1^2}{2^2} + \dfrac{1^2 3^2}{2^2.4^2} + \dfrac{1^2.3^2.5^2}{2^2.4^2.6^2} + \ldots\ldots\infty$.

Ans. Here, $$u_n = \frac{1^2.3^2.5^2\ldots\ldots(2n-1)^2}{2^2.4^2.6^2\ldots\ldots.(2n)^2}$$

$$u_{n+1} = \frac{1^2.3^2.5^2\ldots\ldots(2n-1)^2(2n+1)^2}{2^2.4^2.6^2\ldots.(2n)^2(2n+2)^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{un^2\left(1+\frac{1}{n}\right)^2}{un^2\left(1+\frac{1}{2n}\right)^2} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}$$

$$\therefore \qquad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1 \text{. Hence, the ratio test fails}$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{(2n+2)^2}{(2n+1)^2}-1\right]$$

$$=n\left[\frac{4n^2+8n+4-\left(4n^2+4n+1\right)}{(2n+1)^2}\right]$$

$$=n\frac{(4n+3)}{(2n+1)^2}=\frac{4n^2+3n}{(2m+1)^2}$$

$$=\frac{1+\frac{3}{4n}}{\left(1+\frac{1}{2n}\right)^2}\to 1 \quad \text{as } n\to\infty$$

$\therefore$ Raabe's test also fails.

When D' Alembert ratio test fails, we can directly apply Gauss test.

Now, $$\frac{u_n}{u_{n+1}}=\frac{(2n+2)^2}{(2n+1)^2}=\frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}=\left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^{-2}$$

$$=\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{2}{2n}+\frac{3}{4n^2}-\ldots\ldots\right)$$

$$=1+\frac{1}{n}+\frac{1}{n^2}\left(1-2+\frac{3}{4}\right)+\ldots\ldots$$

$$=1+\frac{1}{n}-\frac{1}{un^2}+\ldots\ldots=1+\frac{1}{n}+0\left(\frac{1}{n^2}\right).$$

Comparing it with $$\frac{u_n}{u_{n+1}}=1+\frac{\lambda}{n}+0\left(\frac{1}{n^2}\right)$$

We have $\lambda=1$. Thus, by Gauss test, the series Σu_n diverges.

Q. 3. (b) For what values of x the series $\frac{1}{1-x}+\frac{1}{2(1-x)^2}+\frac{1}{3(1-x)^3}+\ldots\ldots\infty$ is convergent.

Ans. Let $$u_n=\Sigma\frac{1}{n(1-x)^n}$$

Then $$u_n = \frac{1}{n(1-x)^n}$$

$$u_{n+1} = \frac{1}{(n+1)^n(1-x)^{n+1}}$$

Then. $$\frac{u_n}{u_{n+1}} = \frac{\frac{1}{n(1-x)^n}}{\frac{1}{(n+1)(1-n)^{n+1}}}$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{n(1-x)^n}.(n+1).(1-x)^{n+1}$$

$$\underset{n\to\infty}{Lt}\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right).(1-x)$$

By D' Alembert ratio test.

Σu_n is convert if
$$1-x>1$$
$$-x>1-1$$
$$-x>0$$
$$\boxed{x<0}$$

and Σu_n is divergent if
$$1-x<1$$
$$-x<0$$
$$\boxed{x>0}$$

Section-B

Q. 4. (a) Reduce the matrix $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ into normal form and hence find its rank.

Ans. Let $$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$ Operating R_{12}

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \quad \text{Operating } C_2 + C_1,\ C_3 + 2C_1,\ C_4 + 4C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 7 \\ 3 & 4 & 9 & 10 \\ 6 & 9 & 12 & 17 \end{bmatrix} \quad \text{Operating } R_2 - 2R_1,\ R_3 - 3R_1,\ R_4 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \quad \text{Operating } R_2 \to R_3,\ R_4 \to 2R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 1 & -6 & -3 \end{bmatrix} \quad \text{Operating } C_3 + 6C_2,\ C_4 + 3C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{Operating } R_3 - 4R_2,\ R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Operating } \frac{1}{33}C_3,\ \frac{1}{22}C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Operating } C_4 - C_3$$

$$\sim\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 0\end{array}\right]=\left[\begin{array}{c|c}I_3 & 0\\ \hline 0 & 0\end{array}\right]$$

Which is the required normal form.

$\therefore$ $P(A) = 3$ **Ans.**

Q. 4. (b) Find the values of a and b for which the equations $x + ay + z = 3$, $x + 2y + 2z = b$, $x + 5y + 3z = 9$ are consistent when will these equations have a unique solution?

Ans. Given System of equations :

$$x + ay + z = 3$$

$$x + 2y + 2z = 6$$

$$x + 5y + 3z = 9$$

Matrix form,

$$\begin{bmatrix}1 & a & 1 & : & 3\\ 1 & 2 & 2 & : & b\\ 1 & 5 & 3 & : & 9\end{bmatrix}$$

Operate $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim\begin{bmatrix}1 & a & 1 & : & 3\\ 0 & 2-a & 1 & : & b-3\\ 0 & 5-a & 2 & : & 6\end{bmatrix} \quad \text{Operate } R_3 \to R_3 - 2R_2$$

$$\sim\begin{bmatrix}1 & a & 1 & : & 3\\ 0 & 2-a & 1 & : & b-3\\ 0 & a+1 & 0 & : & 12-2b\end{bmatrix}$$

Now we are given that system of equations have a unique solution so,

$$P(A) = P([A.B]) = \text{Number of variables.}$$

Then $a+1 \neq 0 \Rightarrow a \neq -1$ b is arbitrary

Then $P(A) = P(A:B) = 3$

Hence, $a \neq -1$ b is arbitrary.

Q. 5. (a) Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix}7 & 2 & -2\\ -6 & -1 & 2\\ 6 & 2 & -1\end{bmatrix}$ and hence find A^{-1}.

Ans. Given,
$$A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

Then characterststs equation of A is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)\left[(1+\lambda)^2 - 4\right] - 2\left[6(1+\lambda) - 12\right] - 2\left[-12 + 6(1+\lambda)\right] = 0$$

$$(7-\lambda)\left[1+\lambda^2+2\lambda-4\right] - 2\left[6+6\lambda-12\right] - 2\left[-12+6+6\lambda\right] = 0$$

$$(7-\lambda)\left(\lambda^2+2\lambda-3\right) - 2\left[6\lambda-6\right] - 2\left[6\lambda-6\right] = 0$$

$$7\lambda^2 + 14\lambda - 21 - \lambda^3 - 2\lambda^2 - 3\lambda - 4(6\lambda - 6) = 0$$

$$\Rightarrow \quad -\lambda^3 + 5\lambda^2 + 14\lambda - 21 + 3\lambda - 24\lambda + 24 = 0$$

$$-\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \qquad \ldots(1)$$

Now by Caylay Hamilton's theorem A must satisfies (1) so

$$A^3 - 5A^2 + 7A - 3I = 0$$

Verification,

Now,
$$A^2 = A.A. = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$$

$$A^3 = A.A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}\begin{bmatrix} 25 & 8 & 8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$7A = 7\begin{bmatrix} 7 & 2 & 2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 49 & 14 & -14 \\ -42 & -7 & 14 \\ 42 & 14 & -7 \end{bmatrix}$$

Now,
$$A^3 - 5A^2 + 7A - 3I$$

$$= \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix} - \begin{bmatrix} 125 & 40 & -40 \\ -120 & -35 & 40 \\ 120 & 40 & -35 \end{bmatrix} + \begin{bmatrix} 49 & 14 & -14 \\ -42 & -7 & 14 \\ 42 & 14 & -7 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$A^3 - 5A^2 + 7A - 3I = 0 \qquad ...(2)$$

Multiplying by A^{-1}

$$A^2 - 5A + 7AA^{-1} - 3A^{-1} = 0$$

$$3A^{-1} = A^2 - 5A + 7I$$

$$= \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - \begin{bmatrix} 35 & 10 & -10 \\ -30 & -5 & 10 \\ 30 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} -10 & -2 & 2 \\ 6 & -2 & -2 \\ -6 & -2 & -2 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

Q. 5. (b) Diagonalise the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ and obtain the modal matrix.

Ans. The characteristic equal of A is.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (-1-\lambda)[-\lambda(2-\lambda)+1]-2[-\lambda+1]-2[-1+(2-\lambda)]=0$

$\Rightarrow \quad \lambda^3-\lambda^2-5\lambda+5=0$ ...(1) [On simplification]

Solving it $\quad \lambda = 1, \pm\sqrt{5}$

$\because \lambda = 1$ is a root of (1) by syntheic division.

$$\begin{array}{c|cccc} 1 & & & & \\ \hline & 1 & -1 & -5 & 5 \\ & 0 & -1 & 0 & -5 \\ \hline & 1 & 0 & -5 & 0 \end{array}$$

$$\lambda^2-5=0$$

$$\lambda=\pm\sqrt{5}$$

Solving it $\quad \lambda = 1, \pm\sqrt{5}$

When $\lambda = 1$, the corresponding eigen vector is given by,

$$-2x_1+2y_1-2z_1=0$$

$$x_1+y_1+z_1=0$$

$$x_1-y_1+z_1=0$$

Solving the last two $\dfrac{x_1}{2}=\dfrac{y_1}{0}=\dfrac{z_1}{-2}$ giving the eigen vector $(1, 0, -1)$.

When $\lambda=\sqrt{5}$, the corresponding eigen vector is given by

$$\left(-1-\sqrt{5}\right)x_1+2y_1-2z_1=0$$

$$x_1+\left(2-\sqrt{5}\right)y_1+z_1=0$$

$$-x_1-y_1-\sqrt{5}z_1=0$$

Solving the last theorem,

$$\frac{x_1}{6-2\sqrt{4}}=\frac{y_1}{-1+\sqrt{5}}=\frac{z_1}{1-\sqrt{5}}$$

$$\frac{x_1}{\left(\sqrt{5}-1\right)^2}=\frac{y_1}{\sqrt{5}-1}=\frac{z_1}{1-\sqrt{5}}$$

Or $\quad \dfrac{x_1}{\sqrt{5}-1}=\dfrac{y_1}{1}=\dfrac{z_1}{-1}$ given the eigen vector $\left(\sqrt{5}-1, 1, -1\right)$.

ily, the eigen vector corresponding to $\lambda = -\sqrt{5}$, is $\left(\sqrt{5}+1, -1, 1\right)$.

$\therefore$ The Modal matrix $$B = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

It is the diagonal matrix, obtained by diagolising A.

Section-C

Q. 6. (a) Using Taylor series, compute the value of sin 31° to four decimal places.

Ans. $$\sin(x+h) = \sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + \ldots\ldots$$

First we prove above result

Let $$f(x+h) = \sin(x+h)$$

Putting $h = 0$ $\qquad f(x) = \sin x$

$$f'(x) = \cos x \qquad f''(x) = -\sin x$$

$$f'''(x) = -\cos x \qquad f^{iv}(x) = \sin x$$

$$f^{v}(x) = \cos x \qquad f^{vi}(x) = -\sin x$$

$\therefore$ $$\sin(x+h) = f(x+h)$$

$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots$$

$$= \sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + \frac{h^4}{4}\sin x$$

$$+ \frac{h^5}{5!}\cos x - \frac{h^6}{6!}\sin x - \ldots\ldots$$

$$= \sin x\left[1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \frac{h^6}{6!} + \ldots\ldots\right] + \cos x\left[\frac{h - h^3}{3!} + \frac{h^5}{5} + \ldots\ldots\right]$$

$$= \sin x \cosh + \cos x \sinh$$

$$\Rightarrow \qquad \sin(x+h) = \sin x + h\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + \dots\dots$$

Put $x = 30°$ $h = i$

Then $$\sin x = \sin 30° = \frac{1}{2}, \ \cos x = \cos 30° = \frac{\sqrt{3}}{2} = \frac{1.732}{2} = .866$$

Also $$h = i = \frac{\pi}{180} \text{ radians} = .01745$$

$$h^2 = .000304, \ h^3 = .000005\dots$$

$$\therefore \qquad \sin 31° = \sin 30°\left[1 - \frac{1}{2}(.01745)^2 + \dots\dots\right]$$

$$+\cos 30°\left[.01745 - \frac{1}{6}(.01745)^3 + \dots\right]$$

$$= .5\left[1 - \frac{1}{2}\times .000304 + \dots\dots\right] + .866\left[.01745 - \frac{1}{6}\times .000005 + \dots\dots\right]$$

$$= .5 - .000076 + .015111 = .515035 = .5150 \text{ approximately.}$$

Q. 6. (b) Find the radius of curvature of the curve $y^2 = x^2\dfrac{(a+x)}{(a-x)}$ at the origin.

Ans. Equation of curve is

$$y^2(a-x) = x^2(a+x) \qquad \dots(1)$$

It passes through the origin. Equating to zero the lowest degree terms, we have $a(y^2 - x^2) = 0$.

$\therefore$ $y \neq \pm x$ are the tangents at origin.

$\therefore$ Newton's method is not applicable.

Let $$y = px + \frac{qx^2}{2!} + \dots\dots$$

Putting this value of y in (1), we get

$$\left(px + \frac{qx^2}{2!} + \dots\dots\dots\right)^2 (a-x) = x^2(a+x) \qquad \dots(2)$$

Equating co-efficient of x^2 on both sides $ap^2 = a$ $\therefore$ $p = \pm 1$

Equating co-effs. of x^3 on both sides,

$$-p^2 + apq = 1 \qquad \dots(3)$$

When p = 1 from (3), $-1+aq=1$ or $q=\frac{2}{a}$

$$\therefore \qquad P \text{ at origin } = \frac{\left(1-p^2\right)^{3/2}}{q} = \frac{(1+1)^{3/2}}{\frac{2}{a}} = 2\sqrt{2}.\frac{a}{2}$$

$$= \sqrt{2}a$$

When p = 1 from (3), $-1-aq=1$ n or $q=-\frac{2}{a}$

$$\therefore \qquad P \text{ at origin } = \frac{\left(1+p^2\right)^{3/2}}{q} = \frac{(1+1)^{3/2}}{-\frac{2}{a}} = -\sqrt{2}a$$

Hence, the radii of curvature at the origin are $\pm\sqrt{2}a$.

Q. 7. (a) Transform the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{xy^2}=0$ into polar co-ordinates.

Ans. Putting $\qquad x = r\cos\theta,\ y = r\sin\theta$

Their $\qquad r^2 = x^2 + y$

and $\qquad \theta = \tan^{-1} y/k$

$$r = \sqrt{x^2+y^2} \qquad\qquad \theta = \tan^{-1} y/x$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{n^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{\partial z}$$

$$|||y\ \frac{\partial r}{\partial y} = \sin\theta$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{r^2}}.\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{\partial\cos\theta}{\partial^2} = \frac{\cos\theta}{\partial}$$

Here u is a composite function of x and y.

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}.\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}.\frac{\partial \theta}{\partial x}=\cos\theta.\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}.\frac{\partial u}{\partial \theta}$$

$$\frac{\partial}{\partial x}(u)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{\partial}.\frac{\partial}{\partial\theta}\right)(\mu)\Rightarrow\frac{\partial}{\partial r}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{\partial}.\frac{\partial}{\partial\theta}\quad ...(1)$$

Also

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}=\frac{\partial r}{\partial y}.\frac{\partial u}{\partial y}+\frac{\partial u}{\partial\theta}.\frac{\partial\theta}{\partial y}=\sin\theta.\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}.\frac{\partial u}{\partial\theta}$$

$$\frac{\partial}{\partial y}(u)=\left(\sin\theta.\frac{\partial}{\partial r}+\frac{\cos\theta}{\partial}.\frac{\partial}{\partial\theta}\right)(u)\Rightarrow\frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{\partial}.\frac{\partial}{\partial\theta}\quad ...(2)$$

Now,

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\left(\cos\theta.\frac{\partial}{\partial r}-\frac{\sin\theta}{\partial}.\frac{\partial}{\partial\theta}\right)\left(\cos\frac{\partial u}{\partial r}-\frac{\sin\theta}{\partial}.\frac{\partial u}{\partial\theta}\right)$$

$$=\cos\theta.\frac{\partial}{\partial r}\left(\cos\theta.\frac{\partial u}{\partial r}-\frac{\sin\theta}{\partial}.\frac{\partial u}{\partial\theta}\right)-\frac{\sin\theta}{r}.\frac{\partial}{\partial\theta}\left(\cos\theta.\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$=\cos\theta\left[\cos\theta.\frac{\partial^2 u}{\partial r^2}-\sin\theta\frac{\partial u}{\partial\theta}\left(-\frac{1}{\partial r}\right)-\frac{\sin\theta}{\partial}.\frac{\partial^2 u}{\partial r\partial\theta}\right]$$

$$-\frac{\sin\theta}{\partial}\left[-\sin\theta.\frac{\partial u}{\partial r}+\cos\theta.\frac{\partial^2 u}{\partial\theta\partial r}-\frac{\cos\theta}{\partial}\frac{\partial u}{\partial\theta}-\frac{\sin\theta}{\partial}m\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$=\cos^2\theta.\frac{\partial^2 u}{\partial r^2}+\frac{2\cos\theta.\sin\theta}{r^2}.\frac{\partial u}{\partial\theta}+\frac{\sin\theta}{r}.\frac{\partial u}{\partial r}-\frac{2\cos\theta.\sin\theta}{\partial r\partial\theta}-+\frac{\sin\theta}{\partial r}.\frac{\partial^2 u}{\partial\theta^2}\quad ...(3)$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{\partial}.\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}.\frac{\partial u}{\partial\theta}\right)$$

$$=\sin\theta.\frac{\partial}{\partial r}\left(\sin\theta.\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial r}+\frac{\cos\theta}{\partial}.\frac{\partial u}{\partial\theta}\right)$$

$$=\sin\theta\left[\sin\frac{\partial^2 u}{\partial r^2}-\frac{\cos\theta}{\partial^2}.\frac{\partial u}{\partial\theta}+\frac{\cos\theta}{\partial}-\frac{\partial^2 u}{\partial r\partial\theta}\right]$$

$$+\frac{\cos\theta}{\partial}\left[\cos\frac{\partial u}{\partial r}+\sin\theta.\frac{\partial^2 u}{\partial\theta\partial r}-\frac{\sin\theta}{r}.\frac{\partial u}{\partial r}+\frac{\cos\theta}{r}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$=\sin^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{2\cos\theta.\sin\theta}{\partial^2}.\frac{\partial u}{\partial\theta}\ \frac{\cos^2\theta}{\partial}.\frac{\partial u}{\partial r}+\frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\cos^2\theta}{\partial^2}.\frac{\partial^2 u}{\partial\theta^2}\quad ..(4)$$

Adding equations (3) and (4)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{\partial}\frac{\partial u}{\partial r}+\frac{1}{\partial^2}.\frac{\partial^2 u}{\partial \theta^2}$$

$\therefore$ $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ transfers into } \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{\partial r}.\frac{\partial^2 u}{\partial \theta^2}=0.$$

Q. 7. (b) Evaluate $\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}dx$.

Ans. Let $$F(a)=\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}dx$$

Diff., partially both sides,

$$F'(a)=\frac{d}{d\alpha}F(\alpha)=\int_0^\infty \frac{\partial}{\partial a}\left[\frac{\tan^{-1} ax}{x(1+x^2)}\right]dx$$

$$=\int_0^\infty \frac{1}{(1+x^2)(1+\alpha^2 x^2)}dx \quad \text{[Resolving integrand into partial fractions]}$$

$$=\frac{1}{1-\alpha^2}\int_0^\infty\left[\frac{1}{1+x^2}-\frac{\alpha^2}{1+\alpha^2x^2}\right]dx$$

$$=\frac{1}{1-\alpha^2}\left[\tan^{-1}x-\alpha.\tan^{-1}(\alpha x)\right]_0^\infty$$

Since $\alpha \geq 0$

$\therefore$ $$F^1(\alpha)=\frac{1}{1-\alpha^2}\left[\tan^{-1}\infty-\alpha.\tan^{-1}\infty-\tan 0+\alpha.\tan^{-1}(0)\right]$$

$$=\frac{1}{1-\alpha^2}\left[\frac{\pi}{2}-\alpha.\frac{\pi}{2}\right]$$

$$=\frac{\pi}{2}$$

$$F'(\alpha)=\frac{\pi}{2}.\frac{1}{1+\alpha}$$

Integrating, we get

$$F(a) = \frac{\pi}{2}\log(1+a) + c \qquad \ldots(2)$$

Where 'C' is the constant of integration.

To find C :

Putting $\alpha = 0$ equations (1) and (2), we get

$$F(0) = \text{and } F(0) = 0 + C = 0$$

$$C = 0$$

Hence, from equation (2), we have

$$F(a) = \frac{\pi}{2}\log(1+a).$$

Section-D

Q. 8. (a) Prove that $\displaystyle\int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$, **$n > -1$**.

Ans. As we know that

$$\int_0^{\pi/2} \sin^p x.\cos^2 x\, dx = \frac{1}{2}\cdot\frac{\Gamma\frac{p+1}{2}\cdot\Gamma\frac{q+1}{2}}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$$

Then,
$$\int_0^{\pi/2} \sin^n\theta\, d\theta = \int_0^{\pi/2} \sin^n\theta.\cos^0\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{\Gamma\frac{n+1}{2}\cdot\Gamma\frac{0+1}{2}}{\Gamma\left(\frac{n+1}{2}+\frac{0+1}{2}\right)}$$

$$= \frac{1}{2}\cdot\frac{\Gamma\frac{n+1}{2}\cdot\Gamma\frac{1}{2}}{\Gamma\left(\frac{n+1}{2}+\frac{1}{2}\right)} = \frac{1}{2}\cdot\frac{\Gamma\frac{n+1}{2}\cdot\sqrt{\pi}}{\Gamma\frac{n+2}{2}}, \quad \left[\Gamma\frac{1}{2} = \sqrt{\pi}\right]$$

Hence,
$$\int_0^{\frac{\pi}{2}} \sin^0\theta\, d\theta = \frac{1}{2}\cdot\frac{\Gamma\frac{n+1}{2}\cdot\sqrt{\pi}}{\Gamma\frac{n+2}{2}}$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\left\lceil \frac{n+1}{2} \right.}{\left\lceil \frac{n+2}{2} \right.}.$$

Q. 8. (b) Find the surface area of the solid generated by revolving the cyclical $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about the tangent at the vertex.

Ans. The equation of the cyloid ax

$$x = a(\theta + \sin\theta), \; y = a(1 - \cos\theta)$$

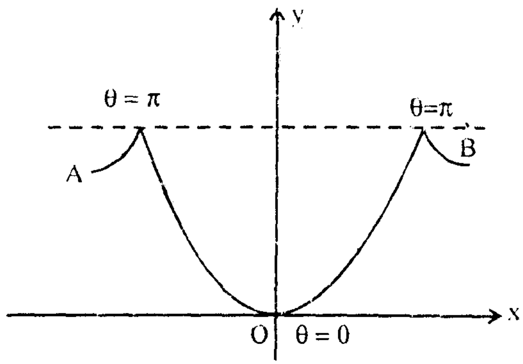
The cyloid is symmetrical about the y-axis and the tangent at the vertex is x-axis. For half of the curve θ. Varies from 0 to π.

Now,
$$\frac{dx}{d\theta} = a(1 + \cos\theta), \; \frac{dy}{d\theta} = a\sin\theta$$

$\therefore$
$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2(1 + \cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{1 + \cos^2\theta + 2\cos\theta + \sin 2\theta} = a.\sqrt{2(1 + \cos\theta)}$$

$$= a.\sqrt{2.2\cos^2\theta/2} = 2a\cos\theta/2$$



$\therefore$ Required surface
$$= 2\int_0^\pi 2\pi y.\frac{ds}{d\theta}d\theta$$

$$= 4\pi\int_0^\pi a(1 - \cos\theta).2\cos\theta/2\,d\theta$$

$$= 4\pi\int_0^\pi a\left(2\sin^2\theta/2\right).2\theta\cos\theta/2\,d\theta$$

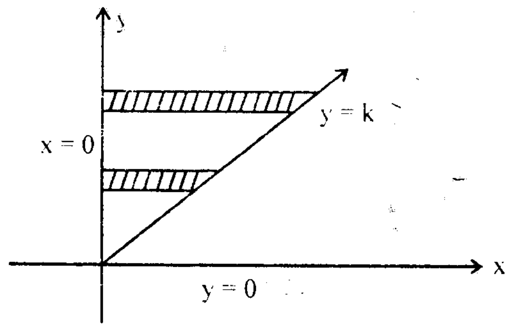
$$= 16\pi a^2 \int_0^{\pi} \sin^2\frac{\theta}{2}.\cos\frac{\theta}{2}\,d\theta$$

$$= 16\pi a^2 \left|\frac{\sin^3\theta/2}{3.\frac{1}{2}}\right|_0^{\pi}$$

$$= \frac{32\pi a^2}{3}.[1-0] = \frac{32}{3}\pi a^3$$

Q. 9. (a) Evaluate $\int_0^{\infty}\int_x^{\infty}\frac{e^{-y}}{y}\,dy\,dx$ by changing the order of integration.

Ans. Let $\int_0^{\infty}\int_x^{\infty}\frac{e^{-y}}{y}\,dy\,dx$



The limits of integration are $0 \le x \le \infty$ and $x \le y \le \infty$. The region of integration is between y-axis and the line $y = x$. In the given order we have to integrate first w.r.t. y and then w.r.t. x. After change of order of we have to integrate first w.r.t. and then w.r.t. y.

Divide the region of integration into horizontal strips. Each horizontal stop starts from line $x = 0$ and end on line $x = y$. Also the strips vary from $y = 0$ and extended to $y = \infty$.

After change of order of integration, we have

$$I = \int_{y=0}^{\infty}\int_{x=0}^{y}\frac{e^{-y}}{y}\,dx\,dy = \int_0^{\infty}\frac{e^{-y}}{y}[x]_0^y.dy$$

$$= \int_0^{\infty}\frac{e^{-y}}{y}.y.dy = \int_0^{\infty}e^{-y}.dy = \left[\frac{e^{-y}}{-1}\right]_0^{\infty} = \left[\frac{e^{-\infty}-e^0}{-1}\right].$$

Q. 9. (b) Evaluate the integral $\iiint\frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$ taken throughout the volume of the sphere $x^2+y^2+z^2=a^2$

Ans. Changing to spherical polar co-ordinates

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta.\sin\phi,\ z = r\cos\phi$$

Then, $V = \{(x,y,z): x^2+y^2+z^2 \le a^2\}$ transforms into

$$V = \{(r,\theta,\phi): 0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi\},\ dx\,dy\,dz = \partial^2 \sin\theta\,dr\,d\theta\,d\phi$$

$$\therefore \quad \iiint_V \frac{dx\,dy\,dz}{\sqrt{a^2 - x^2 - y^2 - z^2}} = \int_0^{2\pi}\int_0^{\pi} \frac{r^2 \sin\theta\,dr\,d\theta\,d\phi}{\sqrt{a^2 - r^2}}$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi/2} \frac{a^2 \sin^2 t}{\sqrt{a^2 - a^2 \sin^2 rt}} \sin\theta\, a\cos t\ dt.d\theta\ d\phi$$

Where $r = a\sin t$

$dr = a\cos t\,dt$

$$= a^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi/2} \sin^2 t\ dt.\sin\theta.d\theta.d\phi$$

$$= a^2 \int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{2}.\frac{\pi}{2}\right]\sin\theta\,d\theta\,d\phi$$

$$= \frac{a^2\pi}{4}\int_0^{2\pi}[-\cos\theta]_0^{\pi}.d\phi$$

$$= -\frac{a^2\pi}{4}\int_0^{2\pi}\left[(-2)d\phi = \frac{a^2\pi}{2}.[\phi]_0^{2\pi} = \frac{a^2\pi}{2}.2\pi\right]$$

$$= a^2\pi^2.$$